

This test covers rotational motion, rotational kinematics, rotational energy, moments of inertia, torque, cross-products, angular momentum and conservation of angular momentum, with some problems requiring a knowledge of basic calculus.

**Part I. Multiple Choice**

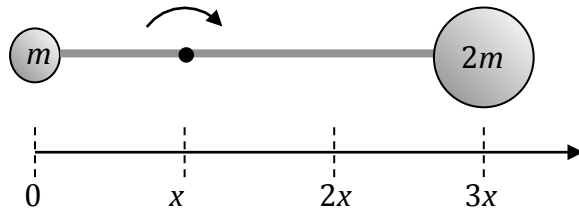
1. A carousel—a horizontal rotating platform—of radius  $r$  is initially at rest, and then begins to accelerate constantly until it has reached an angular velocity  $\omega$  after 2 complete revolutions. What is the angular acceleration of the carousel during this time?

- a.  $\frac{\omega^2}{8\pi}$
- b.  $\frac{\omega^2}{4\pi}$
- c.  $\frac{\omega}{4\pi}$
- d.  $\frac{\omega^2}{4\pi r}$
- e.  $\frac{\omega^2}{2\pi r}$

2. A horizontally-mounted disk with moment of inertia  $I$  spins about a frictionless axle. At time  $t=0$ , the initial angular speed of the disk is  $\omega$ . A constant torque  $\tau$  is applied to the disk, causing it to come to a halt in time  $t$ . How much Power is required to dissipate the wheel's energy during this time?

- a.  $\frac{I\omega^2}{2t}$
- b.  $\frac{I\omega}{t}$
- c.  $\frac{I\omega^2}{2\tau}$
- d.  $\frac{I\omega}{\tau}$
- e.  $\frac{I\omega^2}{2}$

3.

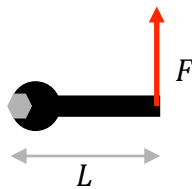


A solid sphere of mass  $m$  is fastened to another sphere of mass  $2m$  by a thin rod with a length of  $3x$ . The spheres have negligible size, and the rod has negligible mass. What is the moment of inertia of the system of spheres as the rod is rotated about the point located at position  $x$ , as shown?

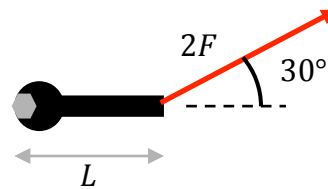
- a.  $3 mx^2$
- b.  $4 mx^2$
- c.  $5 mx^2$
- d.  $9 mx^2$
- e.  $10 mx^2$

4. A series of wrenches of different lengths is used on a hexagonal bolt, as shown below. Which combination of wrench length and Force applies the greatest torque to the bolt?

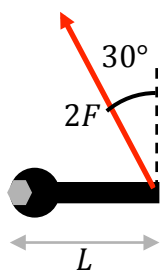
a.



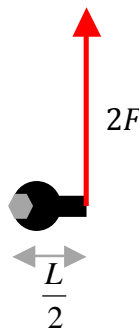
b.



c.



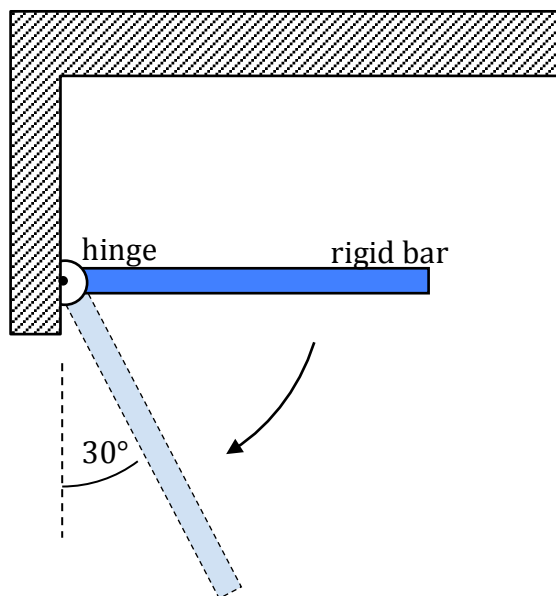
d.



e.



5.

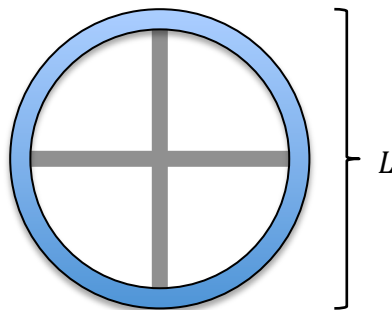


A rigid bar with a mass  $M$  and length  $L$  is free to rotate about a frictionless hinge at a wall. The bar has a moment of inertia  $I = 1/3 ML^2$  about the hinge, and is released from rest when it is in a horizontal position as shown. What is the instantaneous angular acceleration when the bar has swung down so that it makes an angle of  $30^\circ$  to the vertical?

- a.  $\frac{g}{3L}$       b.  $\frac{2g}{3L}$       c.  $\frac{g}{L}$       d.  $\frac{3g}{4L}$       e.  $\frac{3g}{2L}$

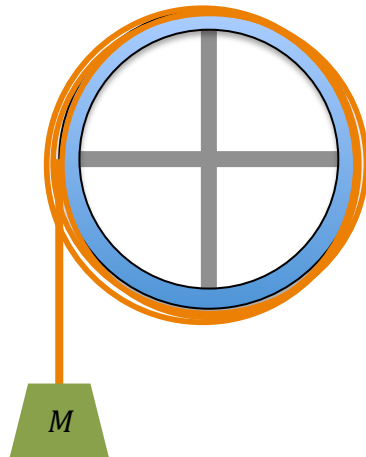
6. A certain star, of mass  $m$  and radius  $r$ , is rotating with a rotational velocity  $\omega$ . After the star collapses, it has the same mass but with a much smaller radius. Which statement below is true?

- a. The star's moment of inertia  $I$  has decreased, and its angular momentum  $L$  has increased.  
b. The star's moment of inertia  $I$  has decreased, and its angular velocity  $\omega$  has decreased.  
c. The star's moment of inertia  $I$  remains constant, and its angular momentum  $L$  has increased.  
d. The star's angular momentum  $L$  remains constant, and its rotational kinetic energy has decreased.  
e. The star's angular momentum  $L$  remains constant, and its rotational kinetic energy has increased.

**Part II. Free Response**

7. A pair of long, thin, rods, each of length  $L$  and mass  $M$ , are connected to a hoop of mass  $M$  and radius  $L/2$  to form a 4-spoked wheel as shown above. Express all answers in terms of the given variables and fundamental constants.

- a. Show that the moment of inertia for one of the rod, for a rotational axis through the center of its length and perpendicular to its length, is  $\frac{1}{12}ML^2$ .
  
  
  
  
  
  
  
  
  
  
- b. Calculate the moment of inertia for the entire spoked-wheel assembly for an axis of rotation through the center of the assembly and perpendicular to the plane of the wheel.



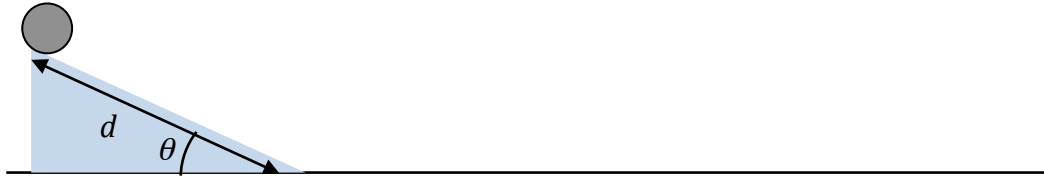
The wheel is now mounted to a frictionless fixed axle and suspended from a vertical support. Several turns of light cord are wrapped around the wheel, and a mass  $M$  is attached to the end of the cord and allowed to hang. The mass is released from rest.

c. Determine the tension in the cord supporting the mass as it accelerates downwards.

d. Calculate the angular acceleration of the wheel as the mass descends.

e. Determine the instantaneous velocity of the mass after the wheel has turned one revolution.

f. Determine the instantaneous angular momentum of the mass-wheel system after the wheel has turned one revolution.

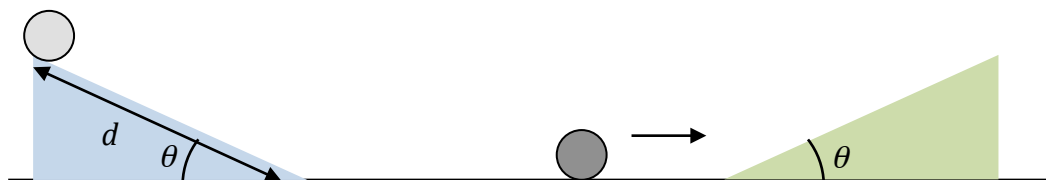


8. A round cylinder has a moment of inertia  $I = \frac{2}{3}MR^2$ , and is released from rest at the top of an incline tilted at  $\theta$  degrees relative to the horizontal. The cylinder rolls down the incline to the bottom, a distance  $d$ , without slipping.
- Draw a free-body diagram of the forces acting on the cylinder, with vectors originating at the point of application.



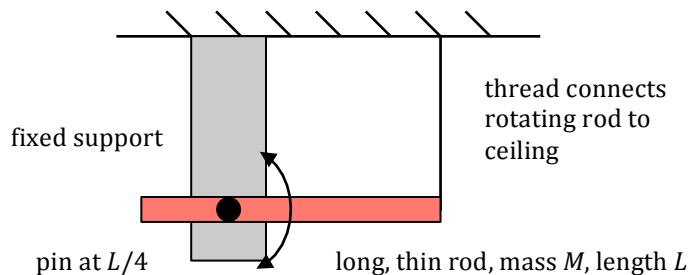
- Determine the acceleration of the cylinder as it rolls down the incline.

- c. Determine the minimum coefficient of friction necessary for the cylinder to be able to roll without slipping down the ramp.



The cylinder rolls along the horizontal surface without slipping, and encounters a second plane inclined at  $\theta$  degrees as shown. The coefficient of friction between this plane and the cylinder is less than the value determined in part (c) above.

- d. Will the maximum vertical height reached by the cylinder on this incline be *more*, *less*, or *the same* as the first one? Explain your reasoning, briefly.
- e. Will the magnitude of the angular acceleration of the cylinder as it rolls up the incline be *more*, *less*, or *the same* as on the first one? Explain your reasoning, briefly.



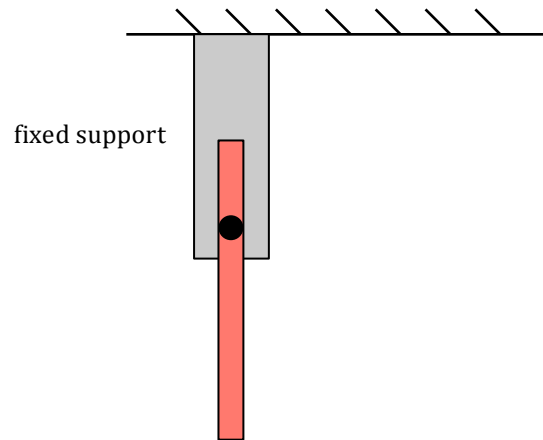
9. A long, thin, rod of mass  $M = 0.500\text{kg}$  and length  $L = 1.00\text{ m}$  is free to pivot about a fixed pin located at  $L/4$ . The rod is held in a horizontal position as shown above by a thread attached to the far right end.

- a. Given that the moment of inertia about an axis of rotation oriented perpendicular to the rod and passing through its center of mass is  $I = \frac{1}{12}ML^2$ , determine the moment of inertia  $I$  of the rod relative to the pivot at  $L/4$ .

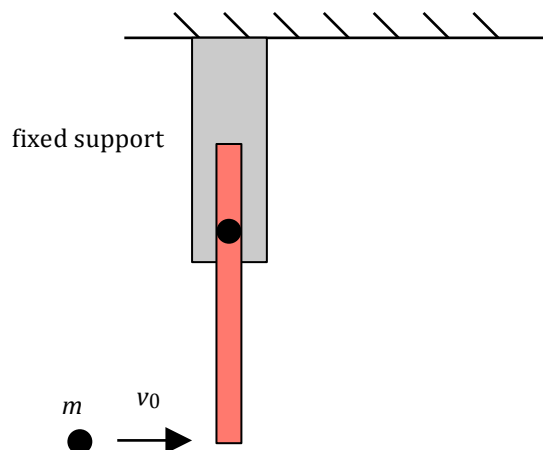
- b. Calculate the tension  $T$  in the thread that supports the rod.

The thread is cut so that the rod is free to pivot about the fixed pin.

- c. Determine the angular acceleration of the rod at the moment the thread is cut.



- d. Determine the angular momentum of the rod relative to the pin at the moment the rod reaches a vertically-oriented position.



Just as the moving rod reaches the vertically-oriented position, it is struck in a head-on elastic collision at the lower end by a ball of mass  $m = 0.500$  kg traveling in a horizontal direction at velocity  $v_0 = 2.00$  m/s as shown.

- e. Determine the velocity, both magnitude and direction, of the ball just after the collision.

1. The correct answer is *a*. The angular acceleration of the carousel can be determined by using rotational kinematics:

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta}$$

$$\alpha = \frac{\omega^2}{2(2 \cdot 2\pi)} = \frac{\omega^2}{8\pi}$$

2. The correct answer is *a*. The Power required to dissipate the wheel's initial energy is calculated using  $P=W/t$ , where  $W$  is the Work required to change the wheel's kinetic energy from its initial value to 0:

$$P = \frac{W}{t}$$

$$W = \Delta K$$

$$\Delta K = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

$$W = 0 - \frac{1}{2}I\omega_0^2$$

$$P = \frac{\frac{1}{2}I\omega_0^2}{t} = \frac{I\omega^2}{2t}$$

3. The correct answer is *d*. Moment of inertia for a system of discrete masses is calculated as follows:

$$I = \sum mr^2$$

$$I = m(x)^2 + 2m(2x)^2$$

$$I = mx^2 + 8mx^2 = 9mx^2$$

4. The correct answer is *c*. Torque, the “turning effect” produced by a Force applied to a moment-arm, is calculated according to  $\tau = \mathbf{r} \times \mathbf{F}$ , or  $\tau = rF \sin \theta$ , where  $\theta$  is the angle between the vectors  $\mathbf{r}$  and  $\mathbf{F}$ . Here, each combination of wrench length and Force produces a net torque of  $LF$  except for answer *c*:

$$\tau = \mathbf{r} \times \mathbf{F} = rF \sin \theta$$

$$\tau = LF \sin 120 = LF \sin 60 = L2F \frac{\sqrt{3}}{2} = LF\sqrt{3}$$

5. The correct answer is *d*. The bar is accelerating angularly in response to the torque due to the force of gravity acting on the center of mass. Its angular acceleration due to this torque at the final position can be calculate as follows:

$$\tau = I\alpha$$

$$r \times F = \left(\frac{1}{3}ML^2\right)\alpha$$

$$\frac{L}{2}mg \sin 30 = \left(\frac{1}{3}ML^2\right)\alpha$$

$$\alpha = \frac{3g}{4L}$$

6. The correct answer is *e*. According to conservation of angular momentum, the angular momentum  $L$  of the star remains constant, so when its moment of inertia  $I$  increases (due to the decreased radius), its angular velocity  $\omega$  goes up proportionally, according to:

$$L_{\text{initial}} = L_{\text{final}}$$

$$I_i\omega_i = I_f\omega_f$$

$$\omega_f = \frac{I_i}{I_f}\omega_i$$

The star's rotational kinetic energy, based on  $K_{\text{rotational}} = \frac{1}{2}I\omega^2$  also goes up. Although  $I$  has decreased,  $K_{\text{rotational}}$  increases with the *square* of  $\omega$ , leading to a net increase in energy.

7.

a. We can determine the moment of inertia for the rod by integrating along its length:

$$I = \int_{-L/2}^{+L/2} r^2 dm$$

$$\lambda = \frac{M}{L}; dm = \lambda d\ell = \lambda dr$$

$$I = \int_{-L/2}^{+L/2} r^2 \lambda dr$$

$$I = \lambda \int_{-L/2}^{+L/2} r^2 dr = \frac{r^3}{3} \Big|_{-L/2}^{+L/2} = \frac{M}{L} \left( \frac{L^3}{8} - \left( \frac{-L^3}{8} \right) \right) = \frac{1}{12} ML^2$$

- b. The moment of inertia for the spoked wheel is simply the sum of the individual moments of inertia of its three components: the two long thin rods and the hoop around the outside:

$$I_{total} = I_{rod} + I_{rod} + I_{hoop}$$

$$I_{total} = \frac{1}{12}ML^2 + \frac{1}{12}ML^2 + MR^2$$

$$I_{total} = \frac{1}{12}ML^2 + \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{5}{12}ML^2$$

- c. To determine the tension in the cord, one approach to solving this problem involves doing Newton's Second Law analyses on the mass and the wheel, and solving for the tension in those two equations.

For the hanging mass:

$$\sum F = ma$$

$$F_g - T = Ma$$

$$Mg - T = Ma$$

For the wheel:

$$\sum \tau = I\alpha$$

$$r \times F = \left(\frac{5}{12}ML^2\right)\left(\frac{a}{r}\right)$$

$$r = \frac{L}{2}, \text{ so}$$

$$\frac{L}{2}T = \left(\frac{5}{12}ML^2\right)\left(\frac{a}{L/2}\right)$$

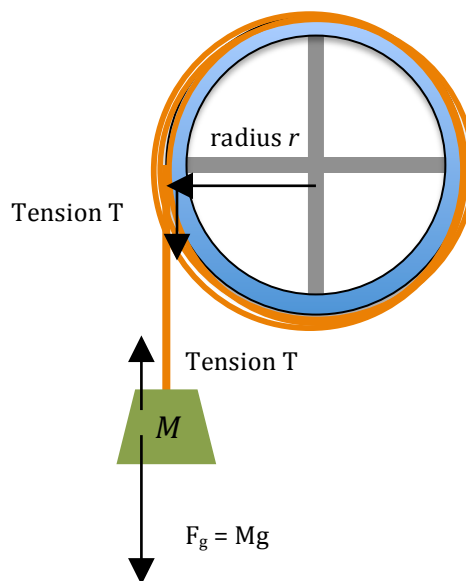
$$a = \frac{3T}{5M}$$

Substitute and solve for  $T$  to get  $T = \frac{5}{8}Mg$ .

- d. Substitute back into equations above to get acceleration, and then angular acceleration:

$$a = \frac{3T}{5M} = \frac{3\left(\frac{5}{8}Mg\right)}{5M} = \frac{3}{8}g$$

$$\alpha = \frac{a}{r} = \frac{\frac{3}{8}g}{L/2} = \frac{3g}{4L}$$



- e. The system is accelerating constantly, so we can use kinematics as follows:

$$v_f^2 = v_i^2 + 2a \Delta x$$

$$v_f = \sqrt{0 + 2\left(\frac{3}{8}g\right)(2\pi r)}$$

$$r = L/2, \text{ so}$$

$$v_f = \sqrt{2\left(\frac{3}{8}g\right)\left(2\pi\frac{L}{2}\right)} = \sqrt{\frac{3}{4}\pi gL}$$

- f. To get the angular momentum of the system, we'll need to include both the moving mass and the rotating wheel:

$$L_{\text{system}} = L_{\text{mass}} + L_{\text{wheel}}$$

$$L_{\text{system}} = r \times mv + I\omega$$

$$L_{\text{system}} = rMv + I\left(\frac{v}{r}\right)$$

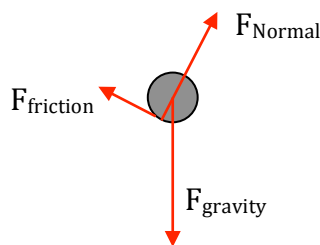
Using answers from previous solutions:

$$L_{\text{system}} = \left(\frac{L}{2}\right)M\sqrt{\frac{3}{4}\pi gL} + \left(\frac{5}{12}ML^2\right)\left(\frac{\sqrt{\frac{3}{4}\pi gL}}{L/2}\right)$$

$$L_{\text{system}} = \frac{4}{3}ML\sqrt{\frac{3}{4}\pi gL} = ML\sqrt{\frac{4}{3}\pi gL}$$

8.

- a. Free-body diagram:



Note that the Normal force should originate from the point where the cylinder touches the surface, as should the force of Friction acting parallel to the plane. The force of gravity should originate at the center of mass of the cylinder.

- b. The acceleration of the cylinder is probably most easily solved by doing an analysis of the Forces and Torques acting on the cylinder, although an energy analysis will eventually lead to the same result as well.

$$\sum F_x = ma_x$$

$$F_{||} - F_{\text{friction}} = ma$$

$$mg \sin \theta = F_{\text{friction}} + ma$$

We have a single equation with both  $F_{\text{friction}}$  and  $a$  unknown. Let's turn to a Torque analysis:

$$\tau = I\alpha$$

$$r \times F_{\text{friction}} = \left(\frac{2}{3}mr^2\right)\left(\frac{a}{r}\right)$$

$$F_{\text{friction}} = \frac{2}{3}ma$$

We can now substitute and solve for acceleration:

$$mg \sin \theta - \frac{2}{3}ma = ma$$

$$a = \frac{3}{5}g \sin \theta$$

- c. To get the minimum coefficient of friction necessary for the object to roll we need to look at the relationship between coefficient of friction  $\mu$ , friction, and the Normal force:

$$\mu = \frac{F_{\text{friction}}}{F_{\text{Normal}}}$$

Here, we can determine the force of Friction acting on the wheel by substituting acceleration back into an equation from answer (b). The Normal force is equal to the perpendicular component of the Force of gravity:

$$F_{\text{friction}} = \frac{2}{3}ma = \frac{2}{3}m\left(\frac{3}{5}g \sin \theta\right) = \frac{2}{5}mg \sin \theta$$

$$F_{\text{Normal}} = mg \cos \theta$$

$$\mu = \frac{F_{\text{friction}}}{F_{\text{Normal}}} = \frac{\frac{2}{5}mg \sin \theta}{mg \cos \theta} = \frac{2}{5} \tan \theta$$

- d. The maximum vertical height reached will be *less* than on the first incline—some of the mechanical energy of the cylinder has been converted to heat. Also, the net Force down the ramp— $F_{\text{gravity}} - F_{\text{friction}}$ —will be greater, so the magnitude of the deceleration is greater, resulting in a shorter travel distance up the ramp.
- e. The magnitude of the angular acceleration of the cylinder as it rolls up the incline will be *less* than it was before. The friction force responsible for torquing the cylinder will be less than it was before, so there is less Torque, and therefore less angular acceleration.
- 9.
- a. We can determine the moment of inertia about this new axis of rotation by using the Parallel Axis Theorem:

$$I = I_{CM} + MD^2$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48}ML^2 = \frac{7}{48}(0.5\text{kg})(1\text{m})^2 = 0.073\text{kg} \cdot \text{m}^2$$

- b. There are a number of ways to solve this, but the easiest is to look at the sum of the Torques about an axis of rotation located at the pivot:

$$\sum \tau = I\alpha$$

$$\tau_{\text{thread}} - \tau_{\text{gravity}} = 0$$

$$\tau_{\text{thread}} = \tau_{\text{gravity}}$$

$$r \times F_{\text{thread}} = r \times F_{\text{gravity}}$$

$$\frac{3}{4}LT = \frac{1}{4}LMg$$

$$T = \frac{1}{3}Mg = \frac{1}{3}(0.5\text{kg})(9.8\text{m/s}^2) = 1.63\text{N}$$

c.

$$\tau = I\alpha$$

$$\alpha = \frac{\tau_{\text{gravity}}}{I} = \frac{r \times F_{\text{gravity}}}{I} = \frac{L/4 Mg}{\frac{7}{48}ML^2} = \frac{12g}{7L} = 16.8\text{rad/s}^2$$

- d. We can easily determine the angular velocity at this new position by using Conservation of Energy. The potential energy of the rod, measured using the height of the center of mass, converts to rotational kinetic energy as the rod rotates down:

$$U_i + K_i + U_f + K_f$$

$$mgh + 0 = 0 + \frac{1}{2}I\omega^2$$

$$mg\left(\frac{L}{4}\right) = \frac{1}{2}\left(\frac{7}{48}ML^2\right)\omega^2$$

$$\omega = \sqrt{\frac{24g}{7L}} = 5.80\text{rad/s}$$

To figure out the angular momentum at this point, then:

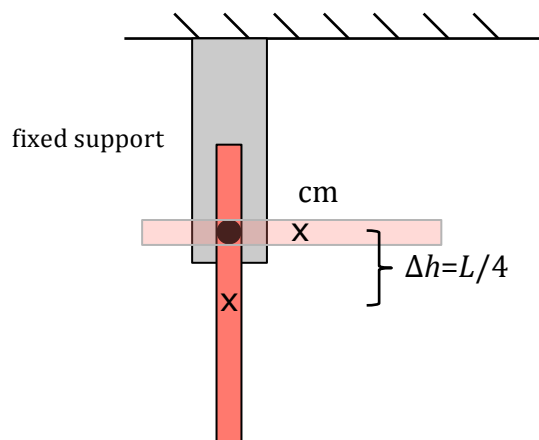
$$L = I\omega$$

$$L = \left(\frac{7}{48}ML^2\right)\omega$$

$$L = \left(\frac{7}{48}(0.5\text{kg})(1.0\text{m})^2\right)5.80\text{rad/s} = 0.423\text{kg}\cdot\text{m}^2/\text{s}$$

- e. This is an elastic collision, so kinetic energy is conserved in the collision, as well as linear momentum and angular momentum. We can solve for the final velocity of the ball (and the rod) after the collision using Conservation of K and Conservation of Angular Momentum.

Let's start with the energy analysis:



$$K_{ball} + K_{rod} = K'_{ball} + K'_{rod}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$$

$$(0.5kg)(2.0m/s)^2 + (0.073kg \cdot m^2)(5.8rad/s)^2 = (0.5kg)v_f^2 + (0.073kg \cdot m^2)\omega_f^2$$

$$61.0 = 6.85v_f^2 + \omega_f^2$$

At this point we have two unknowns, so let's turn to Conservation of Angular Momentum to get another equation with those two unknowns. We'll describe the angular momentum  $L$  of both the rod and the ball relative to the rod's axis of rotation.

$$L_{ball} + -L_{rod} = L'_{ball} + L'_{rod}$$

$$r \times mv_i + I\omega_i = r \times mv_f + I\omega_f$$

$$(0.75m)(0.5kg)(2.0m/s) + -(0.073kg \cdot m^2)(5.80rad/s) =$$

$$(0.75m)(0.5kg)v_f + (0.073kg \cdot m^2)\omega_f$$

$$4.47 - 5.14v_f = \omega_f$$

At this point we have two expressions, both with the same unknown variables. Substitute in to get a quadratic equation that can be solved to get  $v_f$ :

$$61.0 = 6.85v_f^2 + \omega_f^2 \quad \text{and} \quad 4.47 - 5.14v_f = \omega_f$$

$$61.0 = 6.85v_f^2 + (4.47 - 5.14v_f)^2$$

$$v_f = \{-0.62m/s, 2.0m/s\}$$

We have two possible solutions—which one is correct? The ball was traveling at 2.0 m/s before it struck the bar, so it can't possibly continue to have that velocity. Therefore, we choose the -0.62m/s as the correct velocity of the ball after the elastic collision with the rod.