Unit 13: Universal Gravitation – Review Packet

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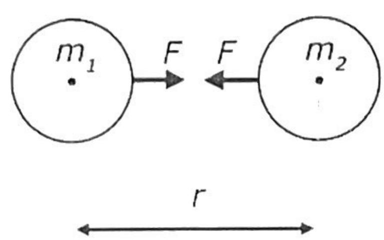
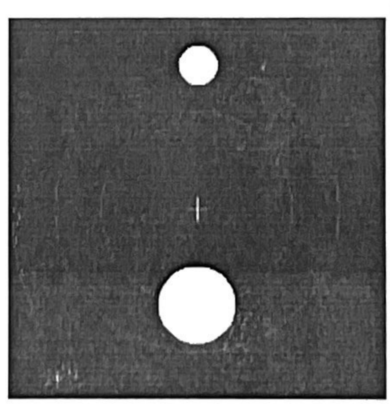
Background:

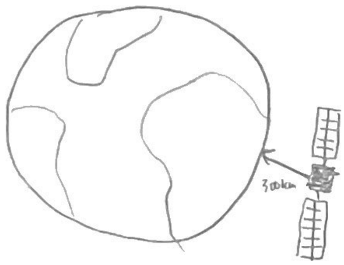
Universal gravitation is the physics behind every single object in the universe attracting every other with gravitational force. The gravitational force is directly dependent on the masses of the two objects and has an inverse relationship with the distance between the two objects.

Topics/Vocabulary	Diagrams	Formulae
Gravitational Force: an attractive force between two objects Gravitational Constant (G): $6.674 \times 10^{-11} \frac{Nm^2}{kg^2}$ Gravitational Potential Energy (U_g): Potential energy of a mass due to its position in a gravitational field Orbital Radius: the distance between the center of a mass and the barycenter Barycenter: the center of mass between 2 objects in an orbital system, and the point they orbit around Weight (W): the force on an object due to gravity	  	$F_G = \frac{Gm_1m_2}{r^2}$ $U_g = mgh$ $U_g = \frac{-Gm_1m_2}{r}$ $W = mg$ $v_{esc} = \sqrt{\frac{2Gm}{r}}$

Let's do some problems!

1. [Easy] What is the gravitational force between Earth (mass 5.97×10^{24}kg, radius 6.37×10^{6} m) and a 1,000 kg satellite 300 km above Earth's surface?
2. [Medium] What is the escape velocity required to get a 500,000kg rocket traveling at 10,000m/s away from the Earth once it reaches the Kármán line?
3. [Hard] What is the orbital radius of the Lagrange point L_1 around the barycenter of Pluto and Charon? Assume $M_{Pluto} = 1.309 \times 10^{22}$ kg, $M_{Charon} = 1.590 \times 10^{21}$ kg, and the distance between the two is 19,640 km.

1)



First find distance from center of Earth:

$r = 6.37e6 + 3.00e5 = 6.67e6$ m

$$F_g = \frac{G m_1 m_2}{r^2}$$

$$= \frac{(6.674e{-11})(5.97e24)(1000)}{(6.67e6)^2}$$

$$= \boxed{8955.89\text{ N}}$$

2)

First perform an energy analysis:

$$U_1 + K_i = U_f + K_f \qquad U_g = \frac{-Gm_1m_2}{r}$$

$$\frac{-GM\cancel{m}}{r_i} + \frac{1}{2}\cancel{m}v_i^2 = \frac{-GM\cancel{m}}{r_f} + \frac{1}{2}\cancel{m}v_f^2$$

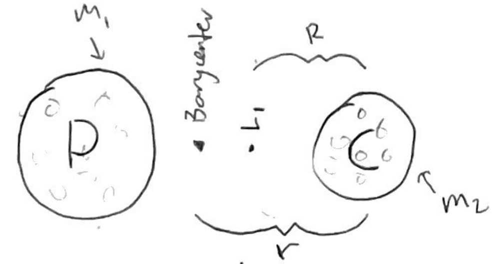
$$v_i^2 = 2\left(\frac{GM}{r_i} - \frac{GM}{r_f} + \frac{1}{2}v_f^2\right)$$

$$v_i = \sqrt{\frac{2GM}{r_i} - \frac{2GM}{r_f} + v_f^2}$$

$$= \sqrt{\frac{2(6.674e\text{-}11)(5.97e24)}{(6.37e6)} - \frac{2(6.674e\text{-}11)(5.97e24)}{(6.37e6 + 1e5)} + 10{,}000^2}$$

$$= \boxed{10096.21 \text{ m/s}}$$

3) First, if we consider circular motion of orbit stemming entirely from gravitation:



$F_g = F_c \to \frac{GM_1 m_2}{r^2} = \frac{m_2 v^2}{r} \to \frac{GM_1}{r} = v^2$

thus to find orbital velocity, we use $v = \sqrt{\frac{GM_1}{r}}$

then using another relationship, $vT = 2\pi r$, where T is the orbital period

with some rearranging, we get $\sqrt{\frac{GM_1}{r}} = \frac{2\pi r}{T} \to \frac{GM_1}{r} = \frac{4\pi^2 r^2}{T^2}$

dividing by r^2, we get $\frac{GM_1}{r^3} = \frac{4\pi^2}{T_2^2}$

then, if we put an object of negligible mass M_3 at point L_1, the forces acting on it are $F = \frac{GM_1M_3}{(r-R^2)} - \frac{GM_2M_3}{R^2}$

now, the centripetal force acting on M_3 must come from gravity. Thus $\frac{GM_1M_3}{(r-R)^2} - \frac{GM_2M_3}{R^2} = \frac{M_3 V_3^2}{(r-R)}$

dividing both sides by $\frac{M_3}{(r-R)}$, we get $\frac{GM_1}{r-R} - \frac{GM_2(r-R)}{R^2} = V_3^2$

if we remember $vT = 2\pi r \to v^2 = \frac{4\pi^2 r^2}{T^2}$, then we substitute in: $\frac{GM_1}{r-R} - \frac{GM_2(r-R)}{R^2} = \frac{4\pi^2 (r-R)^2}{T_3^2}$

dividing by $(r-R)^2$, we get $\frac{GM_1}{(r-R)^3} - \frac{GM_2}{R^2(r-R)} = \frac{4\pi^2}{T_3^2}$

by definition, at L_1, the orbital period of m_3 = the orbital period of m_2, or $T_2 = T_3$

recall $\frac{GM_1}{r^3} = \frac{4\pi^2}{T_2^2}$, since $T_2 = T_3$, we substitute: $\frac{GM_1}{(r-R)^3} - \frac{GM_2}{R^2(r-R)} = \frac{GM_1}{r^3}$

dividing all sides by GM_1, we get $\frac{1}{(r-R)^3} - \frac{M_2}{R^2(r-R)M_1} = \frac{1}{r^3}$. setting $a = \frac{M_2}{M_1}$, we get:

$\frac{1}{(r-R)^3} - \frac{a}{R^2(r-R)} = \frac{1}{r^3}$. multiplying by r^3, we get $\frac{r^3}{(r-R)^3} - \frac{ar^3}{R^2(r-R)} = 1$

now we find r by finding the barycenter of the system:

$r_p = d \cdot \frac{m_2}{m_1 + m_2}$ where r_p = the distance of the barycenter from pluto and d is the distance between Pluto and Charon

$r_p = 1.964e7 \cdot \frac{1.590e21}{1.309e22 + 1.590e21} = 2127220.71$ m. knowing $r = d - r_p$, $r = 1.964e7 - 2.127e6$

$r = 1.751e7$ m. substituting in, we get $\frac{(1.751e7)^3}{(1.751e7-R)^3} - \frac{\left(\frac{1.596e21}{1.309e22}\right)(1.751e7)^3}{(1.751e7-R)(R^2)} = 1$

here, we are forced to use a numeric solver to find R, which returns $R = 23154.5$ m

the question asks us to find the orbital radius of L_1 around the barycenter, which we know from the diagram is $r - R$.

$r - R = 1.751e7 - 2.315e4$

$= \boxed{17486950 \text{ m}}$