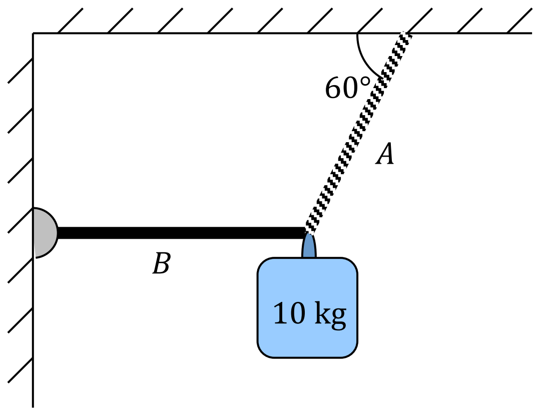
This test covers static equilibrium, universal gravitation, and simple harmonic motion, with some problems requiring a knowledge of basic calculus.

Part I. Multiple Choice

1.



A mass of 10 kg is suspended from a cable *A* and a light, rigid, horizontal bar *B* that is free to rotate, as shown. What is the approximate tension, in Newtons, in cable *A*?

a. $\frac{100}{\sqrt{3}}$

b. $100\sqrt{3}$

c. $\frac{200}{\sqrt{2}}$

d. $200\sqrt{3}$

e. $\frac{200}{\sqrt{3}}$

2. In a science fiction story, a planet has half the radius of the Earth, but the same mass as the earth. What is the acceleration due to gravity at the surface of this planet as a function of **g**?

a. 4**g**

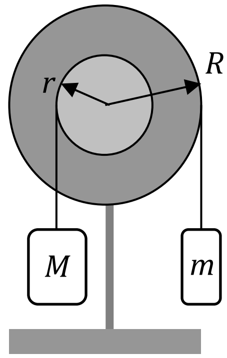
b. 2**g**

c. **g**

d. $\frac{1}{2}$**g**

e. $\frac{1}{4}$**g**

3.



The pulley system consists of two solid disks of different radii fastened together coaxially, with two different masses connected to the pulleys as shown above. Under what condition will this pulley system be in static equilibrium?

a. $m = M$
b. $rm = RM$
c. $r^2m = R^2M$
d. $rM = Rm$
e. $r^2M = R^2m$

4. Earth and Jupiter both travel in a roughly circular orbit around the sun. Jupiter's orbit is approximately 5 times the radius of the Earth's orbit. What is the approximate relationship between the centripetal acceleration of each planet?

a. $a_{Earth} = a_{Jupiter}$
b. $a_{Earth} = 5\ a_{Jupiter}$
c. $a_{Earth} = 25\ a_{Jupiter}$
d. $a_{Jupiter} = 5\ a_{Earth}$
e. $a_{Jupiter} = 25\ a_{Earth}$

5. A simple harmonic oscillator consisting of a mass M attached to a spring with spring constant k is set into motion at the surface of the earth, and observed to have a frequency f. The same spring is then attached to a mass of $2M$, and moved to a location R above the surface of the earth, where R is the radius of the earth. What is the frequency of oscillation now?

a. f
b. $2f$
c. $4f$
d. $f/2$
e. $f/\sqrt{2}$

6. A particle moves constantly in a circle centered at the origin with a period of 4.0 seconds. If its position at time t = 0 seconds is (2,0) meters, two possible equations describing the particle's x- and y-components are:

a. $x = 2\cos\left(\frac{\pi}{2}t\right)$ $y = 2\sin\left(\frac{\pi}{2}t\right)$

b. $x = 2\cos\left(\frac{2}{\pi}t\right)$ $y = 2\sin\left(\frac{2}{\pi}t\right)$

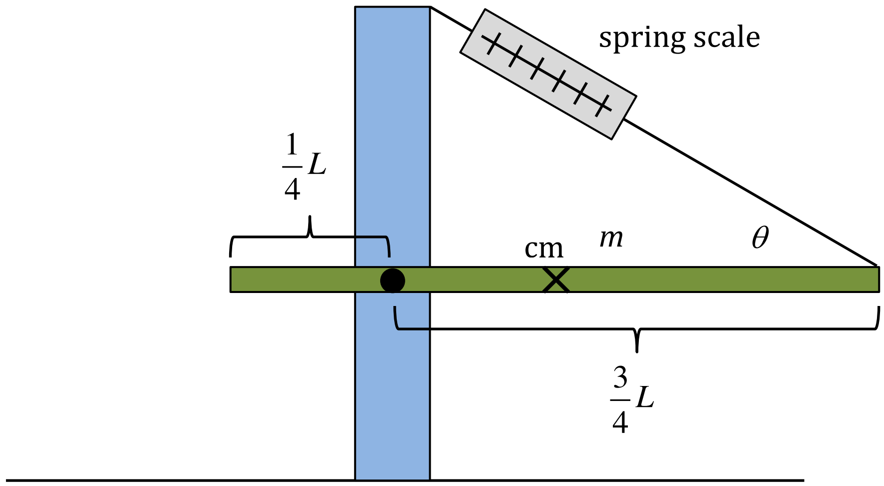
c. $x = 2\sin\left(\frac{\pi}{2}t\right)$ $y = 2\cos\left(\frac{\pi}{2}t\right)$

d. $x = 2\sin\left(\frac{\pi}{2}\right)$ $y = 2\cos\left(\frac{\pi}{2}\right)$

e. $x = 2\pi\cos(2t)$ $y = 2\pi\sin(2t)$

Part II. Free Response

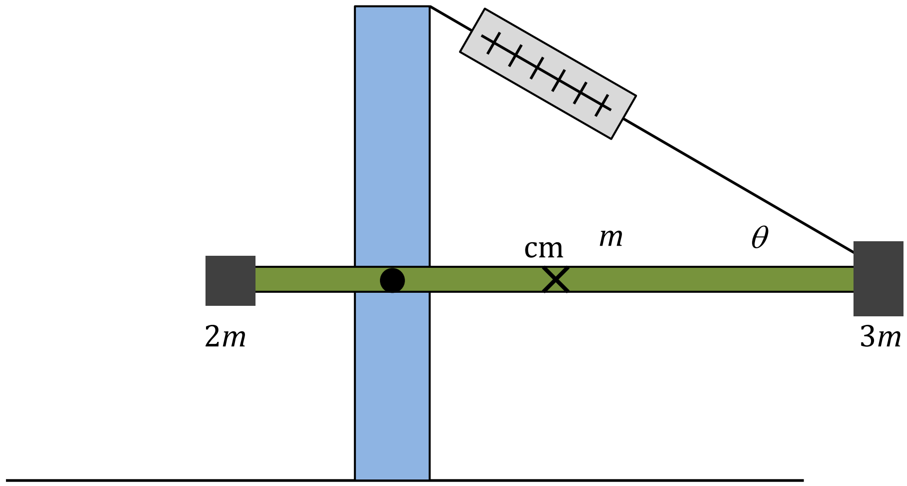
7.



A long, uniform beam with mass *m* and length *L* is attached by means of a pivot, located at L/4, to a vertical support as shown above. The beam is connected to a support line oriented at an angle θ relative to the horizontal; the tension in the support line is indicated by a Force sensor of negligible mass. The beam is currently oriented in a horizontal position. Give all answers in terms of variables given and fundamental constants.

a. Show that the moment of inertia for the rod about the pivot is $\frac{7}{48}ML^2$.

b. Draw a free-body diagram of the horizontal beam.



A weight of mass $2m$ is now firmly attached to the left end of the beam, and a mass of $3m$ attached to the right end of the beam.

c. Calculate the tension T in the support line.

d. Calculate the x and y components of force acting on the pivot point.

e. The support line for the beam is now cut. Calculate the angular acceleration of the beam at the moment the support line is cut.

8. A satellite with mass $m = 617kg$ is placed into a circular orbit $1.00 \times 10^{7} m$ above the surface of the earth, which has a mass $M = 5.98 \times 10^{24} kg$ and a radius $r = 6.38 \times 10^{6} m$.

a. Calculate the linear speed of the satellite as it orbits the earth at this altitude.

b. Calculate the radial acceleration of the satellite.

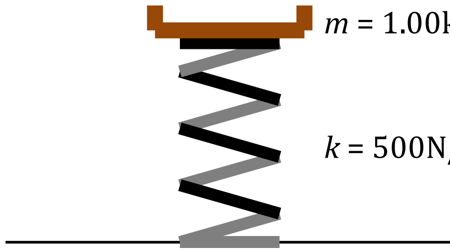
c. Calculate the period of the satellite.

d. Calculate the satellite's total mechanical energy at this altitude.

e. Determine the velocity that was necessary for the satellite to be launched into this orbit from the surface of the earth.

f. Calculate the location of the center-of-mass of the earth-satellite system, relative to the center of the earth.

9.



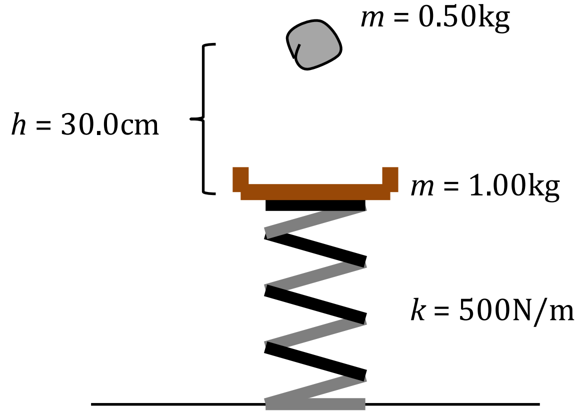
A spring of negligible mass and spring constant 500 N/m is oriented vertically as shown. A flat 1.00 kg platform is then attached to the top of the spring.

a. What distance does the spring compress when the 1.00 kg platform is attached and slowly allowed to descend to an equilibrium position?

The platform is now pulled down an additional 10 centimeters from the equilibrium position and released.

b. Calculate the period of the mass-spring system's oscillation.

c. Determine the speed of the platform as it passes the equilibrium position during its oscillation.



Now the platform is placed again at its equilibrium position. A 500 gram blob of clay is released from a height of 30.0 centimeters above the platform so that it falls and contacts the platform in a perfectly inelastic collision.

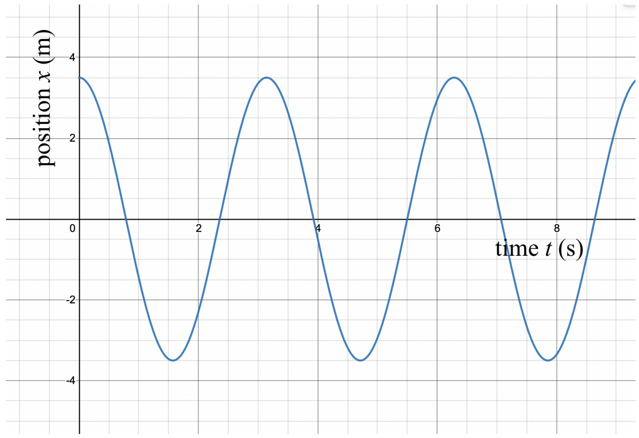
d. Calculate the period of this new mass-spring system.

e. Calculate the amplitude of this new mass-spring system.

The same experiment is now conducted by astronauts at a base station on Mars, where the acceleration due to gravity is less than it is on earth. The 500-gram blob of clay is again released from a height of 30.0 centimeters above the platform, and it falls and collides perfectly inelastically with the platform as before.

f. Compared with the experiment on earth, the period of the mass-spring system on Mars is *more*, *less*, or *the same?* Explain.

g. Compared with the experiment on earth, the amplitude of the mass-spring system on Mars is *more*, *less*, or *the same?* Explain.



10. The period motion of a simple harmonic oscillator is described by the *position-time* graph shown here, with position x given in meters and time t given in seconds.

    a. Determine the amplitude of the oscillator's motion.

    b. Determine the period of the oscillator's motion.

    c. Determine the frequency of the oscillator's motion.

    d. Determine the angular frequency ω of the oscillator's motion.

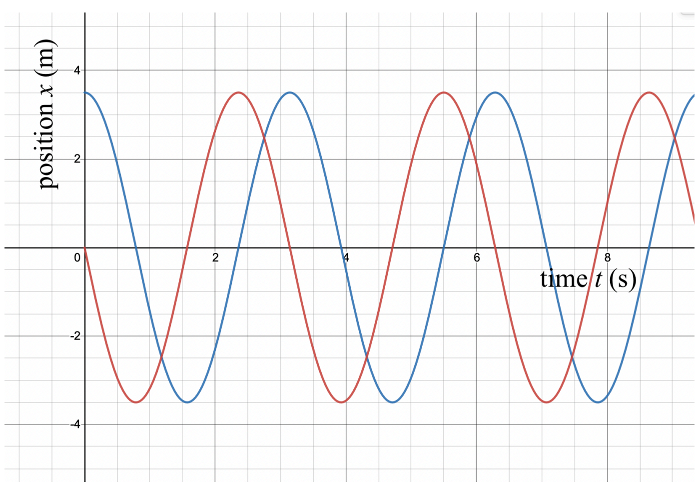
    e. Write an equation that represents the position x of this simple harmonic oscillator as a function of time t, angular frequency ω, amplitude A, and fundamental constants as needed.

The periodic motion in the graph is that of a 2.0 kg mass hung from an ideal spring, set in motion by lifting the mass up a distance from the equilibrium position and releasing it at time $t = 0$.

f. Calculate the spring constant k of the spring.

g. Calculate the total energy of the mass-spring system.

h. Calculate the speed of the oscillating mass as it passes the equilibrium position.



The same oscillator is set into motion, with time $t = 0$ when the mass is moving downwards and at the equilibrium position, as shown by the red line here.

i. Modify the equation from part (e) to correctly represent the oscillator's new position as a function of time.

1. The correct answer is ***e***. The weight of the mass, approximately 100N, must be entirely supported by the vertical component of the tension in the cable, F_y. Therefore:

$$\sum F_y = ma$$
$$F_y - F_g = 0$$
$$F_y = F_{Tension} \sin 60 = (10kg)(\sim 10m/s^2)$$
$$F_{Tension} = \frac{100N}{\sqrt{3}/2} = 200/\sqrt{3}$$

2. The correct answer is ***a***. The acceleration due to gravity can be determined using Newton's Law of Universal Gravitation:

$$F_g = G\frac{m_1 m_2}{r^2}$$
$$m_1\mathbf{g} = G\frac{m_1 m_2}{r^2}$$
$$\mathbf{g} = \frac{Gm_{earth}}{r^2}$$

Now, determine a_{gravity} for the new planet:

$$a_g = \frac{Gm_{planet}}{r_{planet}^2} = \frac{Gm_{earth}}{\left(\frac{1}{2}r_{earth}\right)^2} = \frac{4Gm_{earth}}{r_{earth}^2}$$
$$a_g = 4\mathbf{g}$$

3. The correct answer is ***d***. The pulley system will be in equilibrium when the sum of the torques acting on the pulleys is 0.

$$\sum \tau = 0$$
$$\tau_M - \tau_m = 0$$
$$\tau = r \times F$$
$$(r \times Mg) - (R \times mg) = 0$$
$$rMg = Rmg$$
$$rM = Rm$$

4. The correct answer is ***c***. The centripetal acceleration of each planet is driven by the force of gravity, and the acceleration of a planet can be calculated as follows:

$$F_g = G\frac{Mm}{r^2}$$
$$F_g = ma_g$$
$$ma_g = G\frac{Mm}{r^2}$$
$$a_g = G\frac{M}{r^2}$$

We can see that acceleration due to gravity is inversely proportional to the square of the radius. Jupiter, with its radius 5 times that of the Earth, has $1/5^2$, or 1/25, the acceleration of the Earth. This relationship is consistent with answer *c*.

5. The correct answer is *e*. The location of the mass-spring system doesn't have any effect on its frequency of oscillation, but the mass 2*M* attached to the spring does:

$$f_{mass-spring} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$f' = \frac{1}{2\pi}\sqrt{\frac{k}{2m}} = \frac{f}{\sqrt{2}}$$

Note that changing the location of a pendulum *does* affect the frequency of its oscillation, according to the equation

$$f_{pendulum} = \frac{1}{2\pi}\sqrt{\frac{L}{g}}$$

6. The correct answer is ***a***. Based on the circular path being centered at the origin and particle's original position at (2,0) meters, we can deduce that the amplitude *A* of the particle's motion is 2 meters. Its period of 4.0 seconds allows us to determine its angular velocity:

$$T = \frac{2\pi}{\omega}$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Equations that describe the components of circular motion are:

$$x = A\cos(\omega t + \phi);\ \ y = A\sin(\omega t + \phi)$$

Here, substituting in the values from above, and with $\phi = 0$, the possible equations are:

$$x = 2\cos\left(\frac{\omega}{2}t\right);\ \ y = A\sin\left(\frac{\omega}{2}t\right)$$

Note that we don't know whether the particle is moving in a counterclockwise or clockwise direction; the *y* equation is one possible solution (for a ccw motion). A clockwise motion would have the same equation describing the *x* component, but the *y* component would be given by $y = -A\sin(\omega t + \phi)$, or

$$y = -2\sin\left(\frac{\pi}{2}t\right).$$

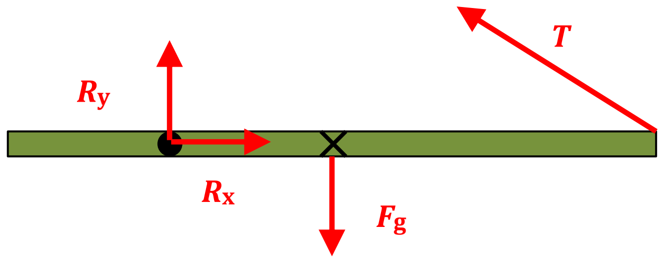
7.

a. There are two ways to arrive at the moment of inertia for the rod: perform the full $I = \int r^2\, dm$ integration with limits (-1/4)L to (+3/4)L, or use the parallel-axis theorem, shown here.

$$I = I_{cm} + MD^2$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48}ML^2$$

b.



c. This is a static equilibrium problem where the sum of the torques acting on the beam is zero.

$$\sum \tau = 0$$

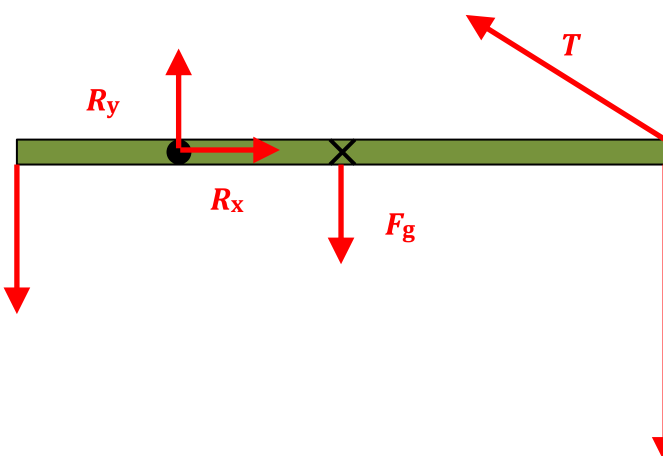
$$\tau_{2m} - \tau_m - \tau_{3m} + \tau_{Tension} = 0$$

$$r \times F_{g2m} - r \times F_g - r \times F_{g3m} + r \times T = 0$$

$$\left(\frac{L}{4}\right)2mg - \left(\frac{L}{4}\right)mg - \left(\frac{3}{4}L\right)3mg + \left(\frac{3}{4}L\right)T\sin\theta = 0$$

$$T = \frac{8mg}{3\sin\theta}$$

d. To determine the R_x and R_y components, we'll use Newton's Second Law in horizontal and vertical directions.



Horizontally:

$$\sum F_x = 0$$

$$R_x - T_x = 0$$

$$R_x = T_x = T\cos\theta = \frac{8mg}{3\tan\theta}$$

Vertically:

$$\sum F_y = 0$$

$$R_y + T_y - F_{2mg} - F_g - F_{3mg} = 0$$

$$R_y = 2mg + mg + 3mg - T\sin\theta$$

$$R_y = 6mg - \left(\frac{8mg}{3\sin\theta}\right)\sin\theta = \frac{10}{3}mg$$

e. To determine angular acceleration, use the angular form of Newton's Second Law of Motion. Note that the moment of inertia of the system has increased with the two masses.

$$I = I_{beam} + I_{2m} + I_{3m} =$$
$$I = I_{beam} + M_2R_2^2 + M_3R_3^2$$
$$I = \frac{7}{48}ML^2 + 2M\left(\frac{L}{4}\right)^2 + 3M\left(\frac{3L}{4}\right)^2$$
$$I = \frac{94}{48}ML^2$$

$$\tau = I\alpha$$
$$+r \times F_{2mg} - r \times F_g - r \times F_{3mg} = I\alpha$$
$$\frac{L}{4}2mg - \frac{L}{4}mg - \frac{3L}{4}3mg = \left(\frac{94}{48}ML^2\right)\alpha$$
$$\alpha = \frac{-96}{94}\frac{g}{L} = \frac{-48}{47}\frac{g}{L}$$

8.

a. The satellite's orbit is maintained by the force of earth's gravity, acting as a centripetal force.

$$F_{gravity} = F_{centripetal}$$
$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$
$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(r_{earth} + altitude)}}$$
$$v = 4.9e3m/s$$

b. The radial acceleration can be calculated in one of two ways:

$$a_c = \frac{v^2}{r} = G\frac{M}{r^2}$$
$$a_c = 1.49m/s^2$$

c. The period of the satellite—the time it takes to complete one orbit—can be calculated as follows:

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$
$$T = \frac{2\pi(6.38e6 + 1.00e7)}{4.9e3} = 2.1e4s = 5.8hrs$$

d. Total mechanical energy is the sum of kinetic and potential energies:

$$E_{total} = K + U$$
$$E_{total} = \frac{1}{2}mv^2 + -G\frac{Mm}{r}$$
$$E_{total} = \frac{1}{2}617(4.9e3)^2 + -(6.672e-11)\frac{(5.98e24)(617)}{(6.38e6 + 1e7)}$$
$$E_{total} = -7.6e9J$$

The shortcut for solving this problem is remembering that, for circular bound orbits,

$$E_{total} = \frac{1}{2}U = -G\frac{Mm}{2R}.$$

e. The satellite has potential energy at the surface of the earth, and needs additional energy (kinetic) to achieve the total energy calculated in part (d).

$$E_{total} = K + U$$

$$-7.6e9J = \frac{1}{2}mv_{esc}^2 + -G\frac{Mm}{r}$$

$$v_{esc} = \sqrt{\frac{2}{m}\left(-7.6e9J + G\frac{Mm}{r}\right)}$$

$$v_{esc} = \sqrt{\frac{2}{617}\left(-7.6e9J + 6.672e-11\frac{(5.98e24)(617)}{6.38e6}\right)}$$

$$v_{esc} = 1.12e4m/s$$

f. The center-of-mass can be calculated using

$$x_{cm} = \frac{x_{earth}m_{earth} + x_{satellite}m_{satellite}}{m_{earth} + m_{satellite}}$$

Considering center of earth as origin:

$$x_{cm} = \frac{0(5.98e24) + 1.638e7)(617)}{5.98e24 + 617} = 1.69e-15m$$

This is obviously *extremely* close to the center of the earth.

9.

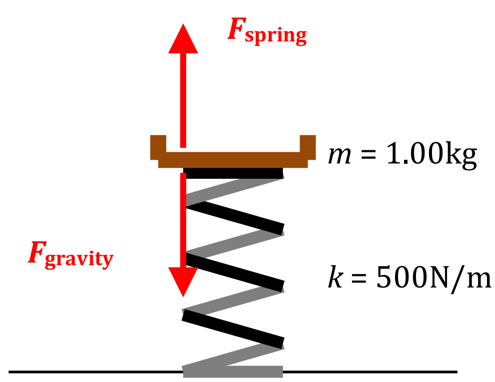
a. When the 1.00kg platform is placed on top of the spring, the spring is compressed until the force of gravity pulling the platform down is balanced by the force of the spring pushing upward. At that point,

$$F_{net} = ma = 0$$

$$+F_{spring} + -F_{gravity} = 0$$

$$kx - mg = 0$$

$$x = \frac{mg}{k} = \frac{(1.0kg)(9.8m/s^2)}{500N/m} = 0.020m$$



b. Calculating period:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\sqrt{\frac{1.00kg}{500N/m}} = 0.28s$$

c. The energy of the platform is based on the stored potential energy when the spring has been compressed 0.10m. At the equilibrium position, that elastic potential energy has been converted to kinetic energy, which allows us to calculate velocity.

$$\sum E_{bottom} = \sum E_{equilibrium}$$
$$U_{elastic} = K$$
$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2$$
$$\frac{1}{2}(500)(0.10)^2 = \frac{1}{2}(1)v^2$$
$$v = 2.24m/s$$

We haven't included the change in potential energy in this analysis because in a vertical spring analysis, the constant force of gravity produces a potential energy term on both sides of the equation that cancel out. (Energy analyses of vertical springs work the same way that horizontal analyses work.)

We can confirm this result by using the harmonic motion equation $v_{\max} = \omega A$.

$$v_{\max} = \omega A$$
$$v_{\max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{500N/m}{1kg}}(0.1m) = 2.24m/s$$

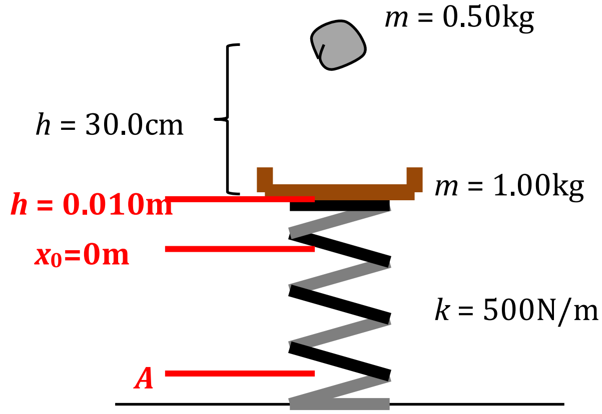
d. Using the same process as before

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$T = 2\pi\sqrt{\frac{1.50kg}{500N/m}} = 0.34s$$

e. The amplitude of the new system is measured as the maximum displacement from the equilibrium position. The new equilibrium position (relative to the unstretched spring, is

$$F_{spring} = F_{gravity}$$
$$kx = mg$$
$$x = \frac{mg}{k} = \frac{(1.5)(9.8)}{500} = 0.0298m = 0.030m$$

We also need to determine the maximum displacement of the spring, which can be calculated based on an energy analysis, where the kinetic energy just after the collision is converted to elastic potential energy.



I'll begin by calculating the velocity of the platform-blob just after the collision, using Conservation of Momentum:

$$p_{blob} + p_{platform} = p_{blob+platform}$$
$$m_{blob}v_{blob} + 0 = (m_{blob} + m_{platform})v'$$

Looks like I need to get the velocity of the blob just before it hits, so I'll do that using energy:

$$U_i + K_i = U_f + K_f$$
$$mgh = \frac{1}{2}mv^2$$
$$v = \sqrt{2gh} = \sqrt{2 \bullet 9.8 \bullet 0.30m} = 2.4m/s$$

Now:

$$p_{blob} + p_{platform} = p_{blob+platform}$$
$$m_{blob}v_{blob} + 0 = (m_{blob} + m_{platform})v'$$
$$(0.5kg)(2.4m/s) = (0.5 + 1.0)v'$$
$$v' = 0.8m/s$$

Now, based on the new equilibrium position, let's calculate the energy changes (discounting changes in gravitational potential energy as before):

$$\sum E_{initial} = \sum E_{final}$$
$$K + U_{spring} = K + U_{spring}$$
$$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2$$
$$\frac{1}{2}(1.5)(0.8)^2 + \frac{1}{2}(500)(0.01)^2 = 0 + \frac{1}{2}(500)A^2$$

Solve equation to get

$$A = \pm 0.0449m$$

f. The period for the oscillating system will be *the same* as it was on Earth. According to the formula for period of a mass-spring system, the only two factors that determine the period are the mass *m* and the spring constant *k*, and those two values are the same on Mars as they were on Earth.

g. The amplitude for the oscillating system will be *less than* it was on Earth. The weaker gravity field will result in a smaller velocity for the blob just before it hits the platform, and less velocity after the collision. With less initial energy in the system, there will be less elastic potential energy stored in the spring, and thus a smaller amplitude when U_{spring} is at a maximum.

a. The amplitude is A = 3.5m, measured from the equilibrium to the peak, or the equilibrium to the trough, of the wave.

b. The period T is the time for one complete cycle, typically identified from measuring peak-to-peak, trough-to-trough, etc. Here, measuring from the peak at time t = 0 to the next peak, the period appears to be a little more than 3… estimate 3.1 s.

c. Calculate frequency as the inverse of the period:
$$f = \frac{1}{T} = \frac{1}{3.1s} = 0.32\ Hz$$

d. Angular frequency ω could be identified by solving $x = A\cos(\omega t + \phi)$ with known values from the graph, but it's probably easier (now that we know period T) to solve this way:
$$T = \frac{2\pi}{\omega}$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.1s} = 2.0\ rad/s$$

e. By inspecting the graph, using the values already identified, and substituting into the displacement equation for simple harmonic motion:
$$x = A\cos(\omega t + \phi)$$
$$x = 3.5\cos(2t + \ \phi)$$
What is the value of ϕ? That *phase constant* is used to identify the position x at time t = 0. In this case, x = 3.5 when t = 0, so the phase constant ϕ is 0. Thus, the equation is
$$x = 3.5\cos(2t)$$

f. For mass-spring oscillators, $\omega = \sqrt{\frac{k}{m}}$. We can determine that by doing a quick derivation of $F_{net} = ma$ with F_{net} supplied by force of the spring, but this analysis is faster:
Period for SHM in general (on equation sheet): $T = \frac{2\pi}{\omega}$
Period for mass-spring system (on equation sheet): $T_S = 2\pi\sqrt{\frac{m}{k}}$
Substitute and solve to see that $\omega = \sqrt{\frac{k}{m}}$, and $k = \omega^2 m$.
$$k = \left(2.0\frac{rad}{s}\right)2.0kg = 8.0N/m$$

g. For a mass-spring system, $E_{total} = U_x + K_x$ for any given position x in the oscillator's motion. At either endpoint of the motion, the mass isn't moving, to all of the energy is stored as potential. Therefore:
$$E_{total} = U_{max} = \frac{1}{2}kA^2$$
$$E_{total} = \frac{1}{2}\left(8.0\frac{N}{m}\right)(3.5m)^2 = 49\ J$$

h. Once we know the total energy of the mass-spring oscillator, we know that that also represents the maximum kinetic energy at the equilibrium position K_0, where the spring is neither compressed nor extended, and therefore momentarily has no potential energy stored in it.

$$E_{total} = K_{max} = \frac{1}{2}mv_0{}^2$$

$$v_0 = \sqrt{\frac{2E_{total}}{m}} = \sqrt{\frac{2(49J)}{2.0kg}} = 7.0\ m/s$$

i. In part (e) we determined that $x = 3.5\cos(2t + \phi)$, with ϕ=0. In this new scenario we can see that *x* has a value of 0 at time *t* = 0. One way of accounting for the new situation is by advancing the function by ¼ of a cycle, or $\frac{\pi}{2}$ radians:

$$x = 3.5\cos\left(2t + \frac{\pi}{2}\right)$$

We could also arrive at the same function by using sin instead of cos, although we'd also need to change the sign of the function to match the graph correctly:

$$x = -3.5\sin(2t)$$